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ON LIMITING NUSSELT NUMBERS FROM MEMBRANE ANALOGY FOR COMBINED FREE AND FORCED CONVECTION THROUGH VERTICAL DUCTS

B. D. AGGARWALA
Department of Mathematics, University of Calgary, Calgary, Canada
M. IQBAL
Department of Mechanical Engineering, University of British Columbia, Vancouver, Canada

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Abstract—Solutions of membrane vibration problems have been utilized for study of fully developed combined free and forced laminar convection through straight vertical triangular ducts of shapes (i) equilateral, (ii) 30°—60°—90° and (iii) right-angled isosceles. Steady state conditions are assumed. All fluid properties are considered constant, except for variation of density in the buoyancy terms. Wall temperature is assumed varying linearly in the flow direction, while it is considered uniform in the transverse direction. Presence of uniform volume heat source has been considered. Exact analytical expressions in the form of infinite series have been presented for velocity, temperature and Nusselt numbers for the three triangular tubes considered. The difference between the Nusselt number for the three triangles becomes relatively small as the value of the Rayleigh number increases. Small values of heat source parameter decreases the Nusselt number.

NOMENCLATURE

A, area of a triangle, dimensionless;
a, characteristic length of a triangle [ft];
a_n, b_n, c_n, d_n, are quantities defined in equations (25), (20), (26) and (22) respectively;
C_p, specific heat of the fluid at constant pressure, [Btu/lb$_m$°F];
C_1, temperature gradient in flow direction [°F/ft];
D_h, hydraulic diameter = (4 · cross-sectional area)/(heat-transfer perimeter [ft];
F, heat generation parameter, dimensionless, $= Q/\rho C_p C_1 U$;
g, gravitational acceleration [ft/hr^2];
h, average peripheral heat transfer coefficient, [BTU/hr ft^2 °F];
L, pressure drop parameter, dimensionless;
m, n, dummy indexes, integers;
N_{NU}, Nusselt number, dimensionless, $= hD_h/k$;
N_{RA}, Rayleigh number, dimensionless, $= (\rho^2 g C_p C_1 \beta D_h^4)/k\mu$;
Q, heat generation rate [Btu/hr ft^3];
q, average surface heat flux [Btu/hr ft^2];
T, temperature [°F];
u, axial velocity [ft/hr];
U, average axial velocity [ft/hr];
v, u/U, dimensionless axial velocity;
x, y, coordinates [ft];
x_1, y_1, dimensionless coordinates;
z, axial coordinate [ft];
θ, $(T - T_w)$ [°F];

ϕ, dimensionless temperature function $= \theta/[\rho U C_p C_1 D_h^2/k]$;
β, ρ, k, μ, are the fluid properties in standard notation;
λ_n, ψ_n, eigen values and eigenfunctions of the equation (15).

INTRODUCTION

IN COMPACT heat exchangers and many other similar installations, ducts of non-circular shape are being employed extensively. For this reason study of fluid flow and heat transfer characteristics in non-circular ducts has become of increasing importance.

It is known that the development of velocity and temperature profiles in a tube depends on the boundary conditions applied on it. From entrance to the heated section of a tube the temperature profile varies all along the tube length when the wall temperature is maintained constant in the axial direction. On the other hand when the wall temperature is allowed to vary at a uniform rate in the axial direction (i.e. uniform heat flux), the temperature profile for fluids of constant properties becomes independent of the axial position after an initial distance of the so-called "entrance length".

In flow through circular tubes, under the condition of uniform axial wall temperature gradient and constant fluid properties, the temperature as well as the temperature gradient along the circumference on the inner surface of a tube remain constant at any axial position of the tube. On the other hand, under the same conditions, for laminar flow through non-circular tubes where flow is retarded at the corners, the temperature gradients along the inner circumference at a section of the tube are expected to be variable [1]. The circumferential wall temperature may, however, tend to be uniform if the tube wall is of high thermal conductivity material and is not too thin.

When we consider laminar free and forced convection flow in a vertical non-circular duct under uniform axial temperature gradient, the above situation of rotationally asymmetric temperature and temperature gradients on the inner surface of the tube wall will also result. The case of vertical circular tube under combined free and forced convection and uniform axial temperature gradient when rotationally symmetric boundary conditions result naturally, has been investigated by Ostroumov [2] and Hallman [3] among others.

For non-circular tubes of certain configuration, when the inner wall temperature at a tube section is assumed constant along the circumference and the axial temperature gradient is uniform, the temperature distribution in the fluid has been obtained from mathematically analogous cases in elasticity. It is known that the equations for deflection of thin plates subjected to uniform lateral load and supported along all edges are identical to the equations describing limiting temperature distributions in fluids in laminar motion in tubes of identical cross-section as those of the plates. Therefore, if the boundary conditions for equation of plate deflection and those for equation of temperature distribution are also identical, then the available solution from plate theory should be readily applicable to a corresponding problem for heat flow. The analogy between equation of fluid flow through a tube and torsion of a uniform rod of same shape has been known since quite some time [4]. Marco and Han [5] have utilized the solution of plate theory to obtain solutions for forced convection heat transfer through non-circular tubes of various configurations. Eckert *et al.* [6] obtained laminar forced convection heat-transfer solutions for flow through wedge shaped passages. In a discussion on a paper by Han [7], Lu [8, 9] has indicated that the problem of combined free and forced convection through vertical rectangular tubes under uniform axial temperature gradient can also be solved by analogy to the plate theory.

The problems in plate theory that bear direct relation to the problem under investigation are (1) simply supported plate lying on an elastic foundation, (2) free vibrations of a simply supported plate and (3) stability problem of a simply supported plate acted on by lateral thrust. Wakasugi [10, 11] has solved the stability problem for various triangular shapes.

In this analysis combined free and forced convection under uniform heat flux of vertical, equilateral, right triangular with 60°, and right-angled isosceles triangular ducts has been investigated.

STATEMENT OF THE PROBLEM AND ASSUMPTIONS

Consider fully developed steady laminar flow through a straight vertical duct of an arbitrary shape as shown in Fig. 1(a). The flow is in the vertical upwards direction along the positive z-axis. Heat flux in the flow direction is considered uniform, while the heat flux in the transverse direction is assumed to vary in such a manner that the wall temperature becomes rotationally symmetrical (i.e. circumferentially uniform wall temperature). All fluid properties are assumed constant except for the variation of density in the body force term of the momentum equation. Viscous dissipation, pressure and axial conduction terms in the energy equation are ignored. The fluid contains uniform volume heat source which is assumed invariant with temperature.

Under the above conditions the wall and the fluid temperature become linear along the flow direction and the governing equations of momentum and energy are as follows,

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \rho g, \tag{1}$$

$$\rho C_p u\frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + Q. \tag{2}$$

The density can be assumed to vary linearly with temperature and for use in the buoyancy terms here, it can be expressed as,

$$\rho = \rho_w[1 - \beta(T - T_w)]. \tag{3}$$

The wall temperature is defined by,

$$T_w = T_0 + z\frac{\partial T}{\partial z},$$

where T_0 is the reference temperature at $z = 0$ and $\partial T/\partial z = C_1, C_1$ being a constant is the temperature gradient in the flow direction. Equations (1) and (2) are to be expressed in non-dimensional terms with the help of the following parameters,

$$\left.\begin{aligned} &x_1 = x/D_h, \qquad y_1 = y/D_h, \\ &v = u/U, \\ &\phi = (T - T_w)\Big/\left[\frac{\rho U C_p C_1 D_h^2}{k}\right], \\ &L = \left[-D_h^2\left(\frac{\partial p}{\partial z} + \rho_w g\right)\right]\Big/\mu U, \\ &F = Q/(\rho C_p C_1 U), \\ &N_{Ra} = (\rho^2 g C_p C_1 \beta D_h^4)/(\mu k). \end{aligned}\right\} \tag{4}$$

Substituting (3) in (1) and using the parameters in (4), the non-dimensionalized equations can be written as,

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial y_1^2} + N_{Ra}\phi = -L, \tag{5}$$

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y_1^2} - v = -F. \tag{6}$$

The equations (5) and (6) are to be solved under the boundary conditions,

$$\phi = v = 0, \quad \text{at the wall.} \tag{7}$$

GENERAL SOLUTION

First of all we try to obtain a general solution of the equations (5)–(7) for the arbitrary shaped duct in Fig. 1(a). We will then specialize the general solution for the three triangular ducts.

Equations (5) and (6) can be combined together to write the resulting equation only in terms of ϕ as,

$$\nabla^4\phi + N_{Ra}\phi = -L, \quad \text{in the duct} \tag{8}$$

with

$$\phi = 0, \quad \nabla^2\phi = -F \quad \text{at the boundary.} \tag{9}$$

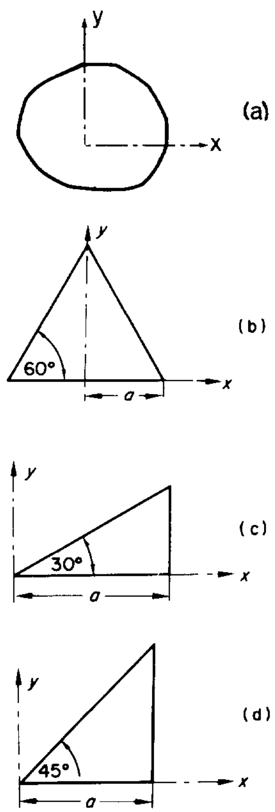

FIG. 1. Coordinate system.

Equation (8) is mathematically equivalent to deflection of a thin plate lying on elastic foundation. Now let

$$\phi = \phi_1 + \phi_2, \tag{10}$$

where

$$\nabla^2\phi_1 = -F, \quad \text{in the duct} \tag{11}$$

with

$$\phi_1 = 0 \quad \text{at the boundary.} \tag{12}$$

This gives

$$\nabla^4\phi_2 + N_{Ra}\phi_2 = -L - N_{Ra}\phi_1 \text{ in the duct} \tag{13}$$

with

$$\phi_2 = \nabla^2\phi_2 = 0 \quad \text{at the boundary.} \tag{14}$$

Our interest is now to reduce the problem to an eigenvalue problem.

Let λ_n and ψ_n, ($n = 1, 2, 3 \ldots \infty$), be the eigenvalues and eigenfunctions of the problem,

$$\nabla^2\psi + \lambda\psi = 0 \quad \text{in the duct} \tag{15}$$

with

$$\psi = 0 \quad \text{at the boundary.} \tag{16}$$

We arrange λ_n in increasing order so that,

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ldots. \tag{17}$$

We assume also that (ψ_n) have been orthonormalized, i.e.

$$\iint \psi_n\psi_m \, dx_1 \, dy_1 = \delta_{mn} \quad \text{for} \quad n \neq m, \tag{18}$$

where δ_{mn} is the Kronecker delta function,

$$\left.\begin{aligned} \delta_{mn} &= 0 \quad \text{for} \quad m \neq n \\ \delta_{mn} &= 1 \quad \text{for} \quad m = n. \end{aligned}\right\} \tag{19}$$

If we now write

$$1 = \sum b_n\psi_n, \tag{20}$$

multiply both sides of this equation by ψ_k and integrate over the duct, we obtain

$$b_k = \iint \psi_k \, dx_1 \, dy_1. \tag{21}$$

It is known [12] that all sufficiently smooth functions can be expanded in terms of ψ_n. We, therefore, write

$$\phi_1 = \sum a_n\psi_n, \tag{21a}$$

$$\phi_2 = \sum C_n\psi_n, \tag{21b}$$

$$\phi = \sum d_n\psi_n, \tag{21c}$$

$$v = \sum g_n\psi_n, \tag{21d}$$

where the constants a_n, c_n, d_n and g_n are to be determined.

From (10) we obtain

$$d_n = a_n + c_n. \tag{22}$$

Also, from (6) we rewrite

$$\begin{aligned} v &= \nabla^2\phi + F\cdot 1 \\ &= \nabla^2\phi + \sum Fb_n\psi_n \end{aligned} \tag{6a}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}.$$

However, from (21c)

$$\nabla^2 \phi = \sum d_n(\nabla^2 \psi_n),$$

and since

$$\nabla^2 \psi_n = -\lambda_n \psi_n,$$

we obtain

$$\nabla^2 \phi = -\sum d_n \lambda_n \psi_n.$$

Substituting this in (6a) gives us the relation for velocity as,

$$\sum g_n \psi_n = -\sum d_n \lambda_n \psi_n + \sum F b_n \psi_n,$$

which gives

$$g_n = F b_n - \lambda_n d_n. \tag{23}$$

Now substituting (21a) in (11), we obtain

$$-\sum a_n \lambda_n \psi_n = -\sum F b_n \psi_n. \tag{24}$$

Therefore,

$$a_n = F b_n / \lambda_n. \tag{25}$$

Similarly, (13) and (14) give

$$c_n = -\frac{L b_n + (F N_{Ra} b_n)/\lambda_n}{\lambda_n^2 + N_{Ra}}. \tag{26}$$

The constants a_n, c_n, d_n and g_n have now been determined.

The pressure drop parameter L is a function of only the Rayleigh number N_{Ra} and the heat generation parameter F. The parameter L is obtained from the continuity equation,

$$\iint v \, \mathrm{d}A = \iint \mathrm{d}A$$
$$= \text{area of the duct, } A, \tag{27}$$

Therefore

$$\sum g_n \left(\iint \psi_n\right) = A.$$

Using (21), this give,

$$\sum g_n b_n = A.$$

Now from (23),

$$\begin{aligned} g_n &= F b_n - \lambda_n d_n \\ &= F b_n - \lambda_n (a_n + c_n) \\ &= F b_n - \lambda_n \left[\frac{F b_n}{\lambda_n} - \frac{L b_n + (F N_{Ra} b_n)/\lambda_n}{\lambda_n^2 + N_{Ra}}\right] \\ &= \frac{L b_n \lambda_n + (F N_{Ra} b_n)}{\lambda_n^2 + N_{Ra}}. \end{aligned}$$

Since, area $= \sum g_n b_n$

$$= L \sum \frac{b_n^2 \lambda_n}{\lambda_n^2 + N_{Ra}} + \sum \frac{F N_{Ra} b_n^2}{(\lambda_n^2 + N_{Ra})}$$

Therefore

$$L = \frac{A - \sum (F N_{Ra} b_n^2)/(\lambda_n^2 + N_{Ra})}{\sum \lambda_n b_n^2/(\lambda_n^2 + N_{Ra})}. \tag{28}$$

This completes the general solution of the problem as far as the velocity and temperature distribution in the duct are required. For engineering purposes one also requires the rate of heat transfer from the wall. An expression for this parameter is developed in the following section.

NUSSELT NUMBER

Nusselt number is the dimensionless parameter indicative of the rate of energy convection from a surface. It is expressed as

$$N_{NU} = \frac{D_h h}{k} = \frac{D_h}{k} \cdot \frac{q}{T_w - T_b} \tag{29}$$

where h, the convection heat-transfer coefficient, is based on mixed mean temperature of the fluid. Nusselt number can be expressed also as,

$$N_{NU} = -\frac{1 - F}{4\,\phi_{mx}}, \tag{30}$$

where ϕ_{mx} is the dimensionless mixed mean temperature difference between the fluid and the wall at the same location,

$$\phi_{mx} = \iint \phi v \, \mathrm{d}A / \iint v \, \mathrm{d}A. \tag{31}$$

Substitution from (21c) and (21d) gives

$$\phi_{mx} = \frac{\sum d_n g_n}{\sum b_n g_n} \tag{32}$$

This completes the general solution of the problem. This means that if we know ψ_n, b_n, and λ_n for a membrane of any shape, the Nusselt number for the ducts of corresponding shape can also be obtained from the same solution. Our interest is to apply the membrane solutions for three duct shapes for which the Nusselt numbers do not appear to be available in the literature.

PARTICULAR SOLUTIONS

We now specialize the general solution to the three triangular ducts. The required quantities ψ_n, b_n, and λ_n for each case are given below.

(1) *Equilateral triangular duct*

The equilateral triangle is shown in Fig. 1(b). It is easy to show that the following expressions for ψ_n are equivalent to those given by Wakasugi [10, equations (5) and (6)].

$$\psi_{mn} = c_{mn}\left[\sin\frac{(2m+n)\pi x}{3a}\sin\frac{n\pi y}{b} - (-1)^{m+n}\sin\frac{(m+2n)\pi x}{3a}\times\sin\frac{m\pi y}{b} + (-1)^m\sin\frac{(m-n)\pi x}{3a}\sin\frac{(m+n)\pi y}{b}\right]$$

$$\bar{\psi}_{mn} = d_{mn}\left[\cos\frac{(2m+n)\pi x}{3a}\sin\frac{n\pi y}{b} + (-1)^{m+n}\cos\frac{(m+2n)\pi x}{3a}\times\sin\frac{m\pi y}{b} + (-1)^{m+1}\cos\frac{(m-n)\pi x}{3a}\sin\frac{(m+n)\pi y}{b}\right]$$

where c_{mn} and d_{mn} are certain constants, and b is the altitude.

In our case, however, b_n is different from zero only for the following eigenfunctions:

$$\psi_n = \frac{\sqrt{2}}{\sqrt{(3ab)}}\left[2\cos\frac{n\pi x}{a}\sin\frac{n\pi y}{b} + (-1)^{n+1}\sin\frac{2n\pi y}{b}\right]. \tag{33}$$

The constant $(\sqrt{2})/\sqrt{(3ab)}$ in ψ_n has been adjusted to satisfy (18). We obtain b_n from (21) and it turns out to be

$$b_n = \frac{(-1)^{n+1}\sqrt{(6ab)}}{n\pi}. \tag{34}$$

Also, we have,

$$\nabla^2\psi_n = \frac{\sqrt{2}}{\sqrt{(3ab)}}\left(-\frac{4n^2\pi^2}{b^2}\right)\left[2\cos\frac{n\pi x}{a}\sin\frac{n\pi y}{b} + (-1)^{n+1}\sin\frac{2n\pi y}{b}\right], \tag{35}$$

therefore

$$\lambda_n = \frac{4n^2\pi^2}{b^2} = \frac{4n^2\pi^2}{3a^2}. \tag{36}$$

(2) 30°, 60° *Right-angled triangular duct*

This triangle is shown in Fig. 1(c). For this triangle the three desired quantities in a similar amount as before following Wakasugi [11] are,

$$\psi_{mn} = \frac{\sqrt{8}}{a\sqrt{(\sqrt{3})}}\left[\sin\frac{m\pi x}{a}\sin\left(\frac{3m+2n}{\sqrt{3}}\cdot\frac{y\pi}{a}\right) - \sin((m+n)\pi x/a)\sin\left(\frac{3m+n}{\sqrt{3}}\cdot\frac{y\pi}{a}\right) + \sin((2m+n)\pi x/a)\sin\frac{n\pi y}{a\sqrt{3}}\right]. \tag{37}$$

$$\lambda_{mn} = \frac{4\pi^2}{3a^2}(3m^2+n^2+3mn). \tag{38}$$

$$b_{mn} = \frac{a\sqrt{25}}{\pi^2\sqrt{(\sqrt{3})}}\left[\frac{1-\cos m\pi}{m(3m+2n)} + \frac{1-\cos n\pi}{n(2m+n)} - \frac{1-\cos(m+n)\pi}{(m+n)(3m+n)}\right]. \tag{39}$$

(3) *Right-angled isosceles triangular duct*

The coordinate system for this triangle is shown in Fig. 1(d) and the required quantities are,

$$\psi_{mn} = \frac{2}{a}\left[\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\right] \quad \text{for} \quad n > m \tag{40}$$

$$b_{mn} = \frac{8an}{m\pi^2(n^2 - m^2)} \tag{41a}$$

if n is even and m is odd $(n > m)$

$$= \frac{8am}{n\pi^2(n^2 - m^2)} \tag{41b}$$

if m is even and n is odd $(n > m)$

$$= 0$$

if n and m are both odd or both even.

$$\lambda_{mn} = (m^2 + n^2)\pi^2/a^2. \tag{42}$$

This completes exact solution of fully developed combined free and forced convection with uniform internal heat sources in the fluid, in the three specified vertical triangular ducts.

DISCUSSION

The values of Nusselt number, equation (30) and the pressure drop parameter, equation (28) have been computed for several values of the Rayleigh number for each of the three triangular configurations. At $F = 0$ those values of Nusselt number and pressure drop parameter are listed in Table 1. At $F = 0$ the variation of Nusselt number and pressure drop parameter against Rayleigh number is also shown in Figs. 2 and 3, respectively. In Fig. 2, Nusselt numbers for only two triangles are shown as there was not enough space on this diagram for the right-angled isosceles. In Fig. 3, one curve is shown as pressure drop parameter for all three ducts, although there is a slight numerical difference in their numerical values, Table 1. At zero Rayleigh number, the values of Nusselt number for the tubes agree with the results of references [13, 14]. At zero Rayleigh number, highest value of the Nusselt number is obtained by the equilateral triangle. As the value of the Rayleigh number is increased, the percentage difference in Nusselt number between the three cases diminishes. It, therefore, means that the presence of large free convection effects tend to

Table 1. Variation of Nusselt number and pressure drop parameter against Rayleigh number for three triangular ducts

Rayleigh number N_{Ra}	Nusselt number N_{NU} 60°—60°—60°	30°—60°—90°	45°—45°—90°	Pressure drop parameter L 60°—60°—60°	30°—60°—90°	45°—45°—90°
0	3·1111	2·8875	2·9819	26·6668	26·0677	26·3063
10	3·1249	2·9067	2·9984	27·4686	26·9307	27·1424
10^2	3·2475	3·0730	3·1436	34·5306	34·4557	34·4690
3×10^2	3·4447	3·3280	3·3719	45·7385	46·1709	45·9797
5×10^2	3·7537	3·7017	3·7183	63·1045	63·9450	63·6066
8×10^2	4·0392	4·0250	4·0275	79·1441	80·1168	79·7410
10^3	4·3029	4·3091	4·3053	94·1270	95·1109	94·7386
2×10^3	5·1761	5·1903	5·1891	146·7824	147·5910	147·2683
3×10^3	5·8375	5·8314	5·8400	192·1201	192·8906	192·5375
5×10^3	6·7971	6·7632	6·7833	271·0402	272·0765	271·5378
8×10^3	7·6444	7·5975	7·6247	357·4286	358·9599	358·1352
10^4	8·2944	8·2406	8·2737	435·7798	437·8083	436·6856
2×10^4	10·0640	9·9778	10·0368	706·7318	710·7794	708·3187
3×10^4	11·2540	11·1304	11·2161	940·8602	947·2163	943·1565
5×10^4	12·9369	12·7397	12·8782	1353·0771	1364·9150	1357·0093
8×10^4	14·4346	14·1474	14·3506	1808·9477	1828·9385	1815·2493
10^5	15·5929	15·2170	15·4837	2224·8617	2254·2112	2233·8457

diminish the difference in heat-transfer rate between the three configurations considered.

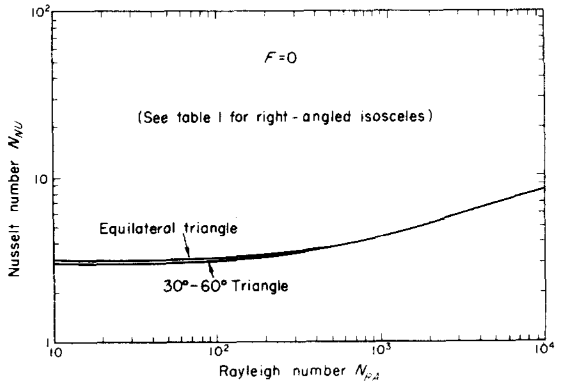

FIG. 2. Nusselt number against Rayleigh number.

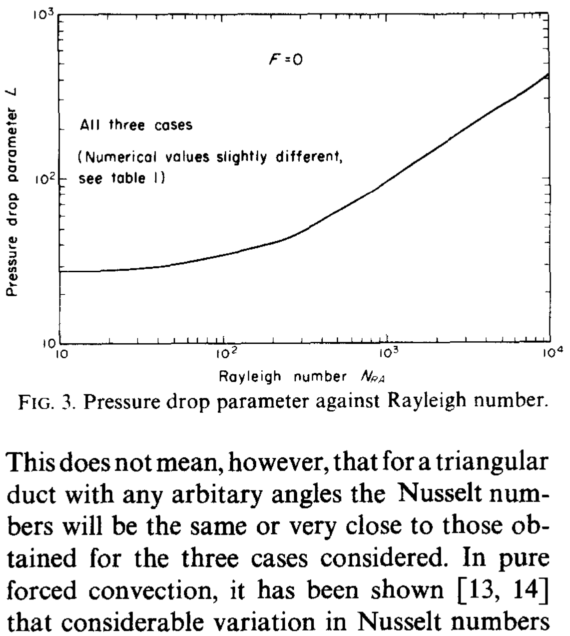

FIG. 3. Pressure drop parameter against Rayleigh number.

This does not mean, however, that for a triangular duct with any arbitary angles the Nusselt numbers will be the same or very close to those obtained for the three cases considered. In pure forced convection, it has been shown [13, 14] that considerable variation in Nusselt numbers results when the angles are varied by large amounts. Unfortunately, the present analysis does not extend itself for any arbitrary angles of a triangular duct. This could have been possible, however, if the corresponding solutions of membrane problems were available.

The velocity profile plots at $F = 0$, are given in Figs. 4–6. Figure 4 is for the equilateral triangle. The velocity profiles are at $x_1 = 0$ and y_1 varying from 0 to 1·5. Three values of the Rayleigh number have been chosen to show significantly different variations. At $x_1 = y_1 = 0$,

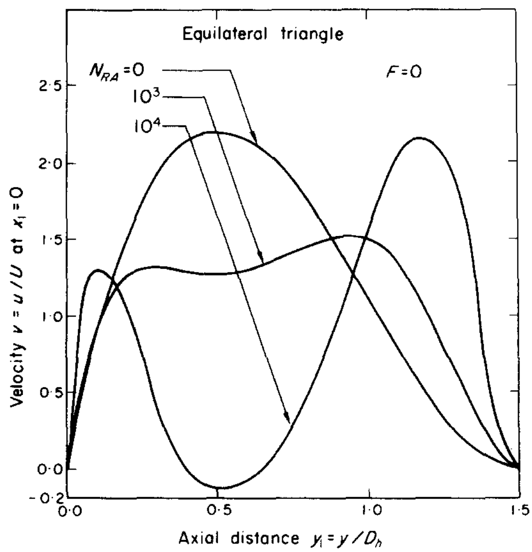

FIG. 4. Dimensionless velocity v against dimensionless axial distance y_1 for various values of the Rayleigh number.

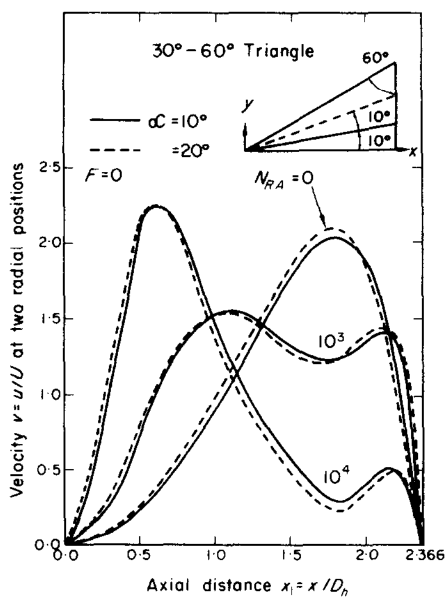

FIG. 5. Dimensionless velocity v at two radial positions against axial distance x_1 for various values of the Rayleigh number.

the velocity gradient is higher than at $x_1 = 0$, $y_1 = 1·5$, for the simple fact that the flow at the corner angles slows down. As the Rayleigh number increases, the flow accelerates near the

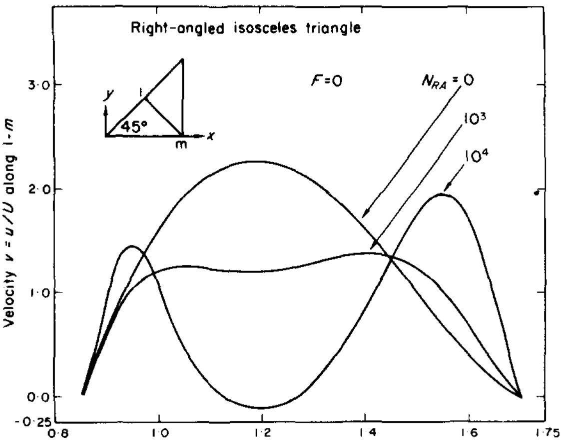

FIG. 6. Dimensionless velocity v along the line $1 - m$ against axial distance x_1 for various values of the Rayleigh number.

walls. To satisfy continuity, the flow slows down near the tube "centre". For large values of the Rayleigh number, it is possible to obtain flow reversal at the tube "centre", while the net flow remains in the upwards direction. This indeed is true at Rayleigh number of 10^4. It is unlikely, however, that in practice a flow reversal near the "centre" will be possible to maintain. For a similar situation in vertical circular tubes it is known [15] that flow reversal at the tube centre or near the wall gives rise to flow instability and eventually turbulent flow results.

The velocity profiles for 30°–60° triangle are shown in Fig. 5. The velocity profiles are plotted for two radial positions; 10° and 20° from the x-axis. They are indicated in solid line and in dotted line respectively. At zero Rayleigh number the velocity gradients at the corner angle of 30° are much smaller than those at the wall, $x_1 = 2{\cdot}366$. As the Rayleigh number is increased, the velocity increases near the walls, while near the "centre" it slows down. The numerical data shows that for this tube, flow reversal will take place at Rayleigh number near to 10^5. This value, however, is not plotted in Fig. 5.

For right angled isosceles triangular tube, the velocity profiles are shown in Fig. 6. These profiles are along a line called $1 - m$, connecting the right-angled corner and middle point of the hypotenuse. These profiles are very similar to those of Fig. 4. The flow reversal occurs at a Rayleigh number of close to 10^4.

At $F = 0$, the temperature profiles for the three triangles are plotted in Figs. 7–9. Figure 7 shows the temperature profiles for the equilateral triangle. As expected, the temperature difference between the wall and fluid decreases as the Rayleigh number increases, This pattern remains the same for the 30°–60° triangle, Fig. 8 and the right-angled isosceles triangle, Fig. 9. This reduced temperature difference with increasing values of the Rayleigh number increases the Nusselt number as is apparent from Table 1 and Fig. 2.

The effect of internal heat generation parameter on various quantities has also been investigated. The heat generation parameter decreases the Nusselt number, until the Nusselt number becomes zero when $F = 1$. This result one expects from physical arguments and also from equation (30). The pressure drop parameter also decreases as the heat generation

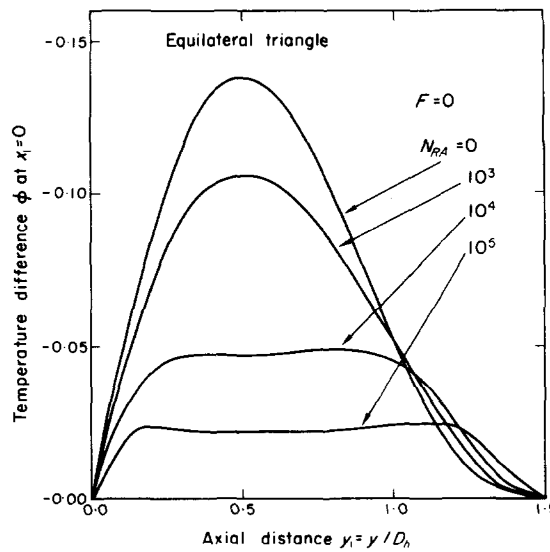

FIG. 7. Dimensionless temperature difference ϕ at $x_1 = 0$ against axial distance y_1 for various values of the Rayleigh number.

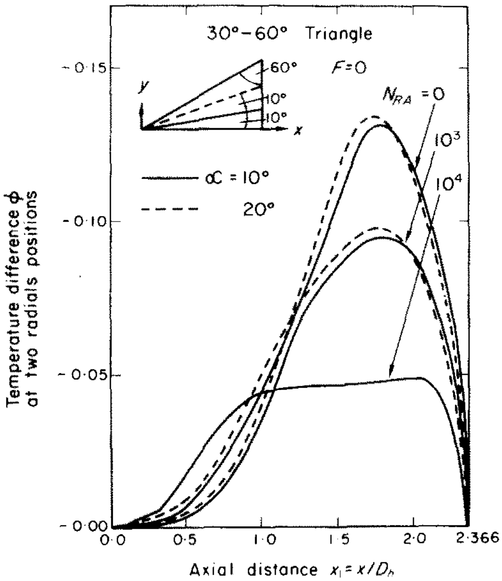

FIG. 8. Dimensionless temperature difference ϕ at two radial positions against axial distance x_1 for various values of the Rayleigh number.

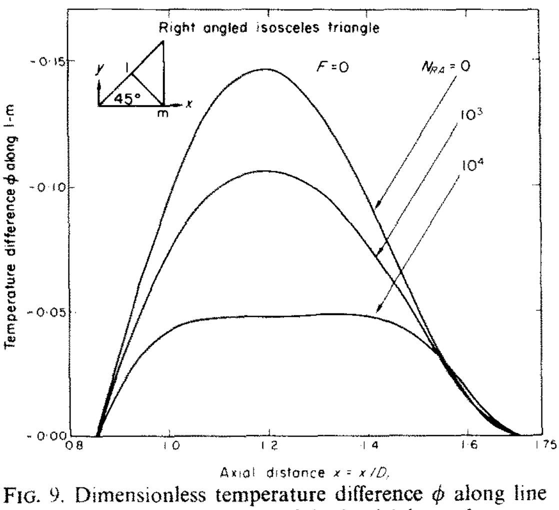

FIG. 9. Dimensionless temperature difference ϕ along line $l - m$ for various values of the Rayleigh number.

parameter increases. However, this happens only when the free convection effect is also present, i.e. when the velocity and temperature fields are coupled. Therefore, the influence of heat generation parameter on the pressure drop parameter depends upon the strength of the coupling of velocity and temperature fields, i.e. on the value of the Rayleigh number as well.

The internal heat generation parameter effects the velocity distribution as well, as long as there is coupling between the velocity and temperature fields, i.e. the presence of free convection in the flow. The heat generation parameter has a dominant effect on the temperature distribution. This effect is independent of the above mentioned coupling, however.

The influence of heat generation parameter on Nusselt number, pressure drop parameter, velocity and temperature distribution in the three triangular ducts has not been illustrated with the help of diagrams. The reason for this is that the general effect of the heat generation parameter is similar to the reports of other researchers, for instance [3] and [7]. From the foregoing analysis, one can indeed evaluate exactly any of the desired quantities with or without heat generation parameter and Rayleigh number.

CONCLUSIONS

Exact analytical solutions of combined free and forced convection through vertical triangular tubes of shapes (i) equilateral, (ii) 30°—60°—90° triangular and (iii) right-angled isosceles triangular tubes have been presented. Effect of heat generation has been included. Nusselt number increases with Rayleigh number. However, the relative difference between the Nusselt number for the three shapes studied diminishes as the Rayleigh number is increased.

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Résumé—Des solutions des problèmes de vibration de membrane ont été utilisées pour l'étude de la convection laminaire mixte (naturelle et forcée) entièrement établie à travers des conduites rectilignes verticales à sections triangulaires 1°) équilatérales, 2°) avec des angles égaux à 30°, 60° et 90° et 3°) isocèle avec un angle droit. On suppose qu'on se trouve en régime permanent. Toutes les propriétés de fluide sont considérées comme constantes, sauf pour la variation de densité dans les termes de poussée d'Archimède. On suppose que la température pariétale varie linéairement dans la direction de l'écoulement, tandis qu'on la considère comme uniforme dans la direction transversale. On a tenu compte de la présence d'une source de chaleur volumique uniforme. Des expressions théoriques exactes sous la forme de séries infinies ont été présentées pour la vitesse, la température et les nombres de Nusselt pour les trois tubes triangulaires considérés. Les différences entre les nombres de Nusselt pour les trois conduites triangulaires deviennent relativement faibles lorsque la valeur du nombre de Rayleigh augmente. Les petites valeurs du paramètre de source de chaleur diminuent le nombre de Nusselt.

Zusammenfassung—Lösungen von Membranschwingungsproblemen wurden zum Studium der vollausgebildeten, freien und erzwurgenen laminaren Konvektion durch gerade, senktechte Dreickskanäle folgender Gestalt herangezogen, (i) gleichseitig, (ii) 30° − 60° − 90° und (iii) rechtwinklig, gleichschenklig.

Stationärer Zustand sei vorausgesetzt. Alle stoffwerte mit Ausnahme der Dichte im Auftriebsglied werden als konstant vorausgesetzt. Die Wandtemperatur ändert sich linear in Strömungsrichtung und ist gleichförmig in Richtung senkrecht dazu.

Gleichförmig verteilte Wärmequellen wurden mit in Betracht gezogen.

Für die drei betrachteten Kanäle werden das Geschwindigkeits- und Temperaturfeld und die Nusseltzahlen durch exakte Ausdrücke in Form unendlicher Reihen dargestellt.

Mit steigenden Werten der Rayleigh–Zahl wird der Unterschied zwischen den Nusselzahlen für die drei Dreiecke relativ klein. Kleine Werte für den Wärmequellenparameter senken die Nusselt–Zahl.

Аннотация—Решения задач для колеблющейся мембраны использовались для изучения полностью развитой совместной свободной и вынужденной ламинарной конвекции в прямых вертикальных треугольных каналах следующих форм :

(1) Равносторонней, (2) 30°–60°–90° и (3) прямоугольной равнобедренной. Принято, что условия являются стационарными. Все свойства жидкости считаются постоянными, за исключением плотности, входящей в члены с подъемной силой. Принято, что температура стенки изменяется линейно в направлении течения, в то время как в поперечном направлении она считается однородной. Рассмотрено наличие однородного объемного

источника тепла. В виде бесконечных рядов представлены точные аналитические выражения для скорости, температуры и критерия Нуссельта для рассмотренных труб трех форм. Разность между значениями критерия Нуссельта для трех типов треугольных каналов становится относительно небольшой по мере увеличения критерия Релея. Небольшие значения параметра теплового источника снижает критерий Нуссельта.